Real Numbers as Relational Tropes: Locating Mathematics in the Causal Order

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1 Can Numbers be Causally Inert?

It is widely assumed that numbers and other mathematical objects, if they exist at all, exist outside spacetime and, therefore, beyond the reach of any causal nexus. As Paul Benacerraf noted, this causal isolation of the numbers raises problems both for our reference to particular numbers in thought and language, and to our capacity to know anything about them. Our best theories of reference and knowledge, thanks to Kripke and Gettier, involve an ineliminable causal connection between names an their referents, and acts of knowledge and their objects.

These assumptions derive from a still deeper set of presuppositions: those of nominalism (the thesis that only particulars exist) and concretism (the thesis that all particulars are concrete). These theses were almost axiomatic for much early analytic philosophy (with the exception of Russell, Bergmann and a few others), but they can't be taken for granted any longer. Recent work on properties, including the revival of theories of universals and tropes, opens up new possibilities for the philosophy of mathematics, possibilities that so far have been little explored. One exception to this is John Bigelow's 1988 book, *The Reality of Numbers: A Physicalist's Philosophy of Mathematics*, that applies an Armstrongian theory of immanent universals. In this paper, I'd like to explore a variant of Bigelow's strategy: considering the possibility that numbers are relational tropes, abstract particulars, and that numerical universals are somehow constituted by or composed of such tropes.

Before getting into the details of this account, let's run through some reasons for being dissatisfied with each of the standard alternatives.

- A Quinean realism about sets and numbers runs into Benacerraf's twin problems, leading to radical indeterminacy of reference and to insoluble Gettier problems for our knowledge of mathematics.
- Gödel's version of Platonism runs into difficulty in explaining the applicability of mathematics to the physical world. In Gödel's view, the numbers

interact with the human mind via a kind of mathematical vision. The main difficulty with this account is that it would seem to explain the applicability of mathematics to the mental sciences, like psychology, linguistics and sociology, but leave the applicability of mathematics to the physical world a complete mystery. In contrast, if we can explain how the numbers are part of the order of physical causation, then we can explain their causal relevance both to the physical and to the mental, assuming, of course, that we can explain the causal relevance of the physical to the mental.

- Structuralism seems all right, so far as it goes, but structuralist owe as an account of what structures are and how we are able to gain knowledge about them. I will argue that structures should be identified with tropes.

- Fictionalism and if-thenism, although attractive as metaphysical theories, have no more success in providing an adequate epistemology of mathematics than does naive realism. In order to know that mathematics is useful, we must know that adding mathematical theory always produces a conservative extension of our scientific theories. We can know this only if we know that the mathematical theories are consistent. However, you can't tell whether a set of axioms is consistent by simply inspecting the symbols, nor by failing to find a contradiction after a finite number of trials. The only way to know that the theory is consistent is to find a true interpretation of its axioms. In the case of mathematical theories, any true interpretation would imply the existence of numbers of some kind. Similarly, relative consistency proofs offer no ultimate solution, since we can only prove that one theory is consistent, relative to an even stronger mathematical theory.

So, let's see if we can locate numbers in an ontological taxonomy of properties. Consider the following questions:

- Are numbers properties of some kind?

- If numbers are properties, what are they properties of?

- If numbers are properties, are they particulars (tropes or individual accidents) or universals?

- If numbers are universals, are they immanent or transcendent? (Do they have multiple spatiotemporal locations, or are they essentially unlocated?)

There is some reason to think of numbers as a property of some kind. A natural number would seem to be some kind of characteristic of a set, extension, plurality, or what-have-you. Real numbers can be used as a measure of quantity, and measures would seem to characterize things.

If we do classify numbers as properties, we are not thereby barred from supposing that they themselves are bearers of further properties. Tropes can stand in various relations to other tropes, for example, as can universals.

If we do classify numbers as properties, we would have the beginnings of a solution to Benacerraf's two problems. Properties do enter into causal relations. In fact, on some views of causation, it is only properties that do see – see, for example, Douglas Ehring's *Causation and Persistence* or Laurie Paul's " Aspect Causation". If numbers are tropes, then they have spatial and temporal location and can straightforwardly enter into a causal nexus with other tropes. If numbers are immanent universals, then they have multiple spatiotemporal locations, and so again are available for causal interaction. Even if numbers are transcendent universals, there are states of affairs that are the instantiation of those universals by particulars, and these states of affairs could serve as the relata of causation.

2 Focusing on Real Numbers

Traditionally, philosophers have focused on either pure sets or the natural numbers as the most fundamental mathematical entities. I'm going to suggest that we consider starting elsewhere: with shapes and with real numbers. There are at least two reasons for starting with real numbers. First, if we can work out the metaphysics of real numbers, we can always extract arithmetic as a subtheory. Second, historically speaking, real analysis is the natural home for set theory. We could perhaps justify the axioms of set theory on the grounds of their fruitfulness in leading to truths about real analysis.

Before we get to real numbers, let's consider shapes for a moment. Shapes are unproblematically properties of physical objects and spatial regions, and as such there is no special problem about giving shapes a causal role. Consider the case of opening a lock with a key. The shape of the key is clearly causally relevant to the key's power to open the lock.

It could be objected that it is only applied or physical geometry, not pure geometry, that consists in the study of physical shapes. If we're going to be serious about finding a causal role for mathematical objects, then this distinction between pure and applied mathematics is something we'll have to challenge. I'm tempted to claim that all mathematics is "applied".

In this spirit, let's turn to the role of real numbers in physics. Real numbers are used as measures of quantities, both extensive (distance, duration) and intensive (mass, charge, energy). A measure is something essentially relational. Consequently, I would propose, following Bigelow, that the fundamental role of real numbers is as relations between physical quantities. Thus, real numbers are relational properties. (More precisely, real numbers are relations between scalar quantities. Relations among vector quantities include the complex numbers.)

Without prejudice against the other alternatives, I am going to assume from now on that real numbers are relational *tropes*, abstract particulars. By talking of 'tropes', I don't mean to be committing myself to a bundle theory of substances. In fact, for the sake of simplicity, I'll assume that there are, in addition to tropes, bare particulars that are the bearers or loci of tropes. That is, I'm assuming a kind of Aristotelian picture, with both primary substances and individual accidents as real entities.

Definition 1 *An event consists in the bearing by one or more entities of a trope (either monadic or relational).*

Definition 2 *A* process *is a four-dimensional object, consisting of a set of interconnected events. Processes are continuous and closed under* interpolation*: that is, if a process P contains events e and e', and these events involve the same particulars at times t and t', then P contains all relevantly similar events involving those same particulars and located between t and t'. A process also contains all relevant tropes of spatial and temporal distance, and all relevant quantitative relational tropes.*

Suppose events e and e' both belong to a process P. Then P will contain a spatial-distance trope that is borne jointly by e and e'. Distance tropes are binary relational tropes. This idea of relational tropes or accidents raises a number of interesting metaphysical and ontological questions, which partly for reasons of time, and partly because they are really hard, I will pass by without attempting to answer.

If process P contains two distinct spatial-distance tropes t and t', then it also contains two real-number tropes borne jointly by the pair t and t', one being the ratio of t to t', and the other the inverse ratio of t' to t. Thus, these real-number tropes are borne somehow by ordered pairs of physical-quantity tropes. A similar story should be told about real-number ratios between temporal distances.

In the case of monadic, conserved quantities. like mass, charge and energy, I think the story has to go somewhat differently. These quantities seem to be intrinsic in a way that spatial and temporal distance are not. That is, there must be some sort of internal structure to a substance that somehow constitutes its having the mass that it has. I can think of two ways of doing this: an atomistic and a Thomistic way. On the atomistic story, there exist atomic mass-tropes, tropes that correspond to a fixed, mininum quantum of mass. The mass of any particle is determined simply by the number of atomic mass-tropes that it bears.

On the Thomistic alternative, there is a absolute maximum quantity of mass, and the mass of particular substance is determined by the number and structure of mass-privations that reduce the mass of a trope to some determinate degree. In either case, there is a natural unit of mass, either a minimum or a maximum. The true mass of a substance would then consist in a real-number trope that relates the quantity of the substance's mass to the quantity of the natural unit. Each substance would in some sense contain the natural unit (either the minimum or maximum) as a kind of a part, and so the quantity of mass would be wholly intrinsic to the substance itself.

Thus, we can give an account of the positive real numbers in this way. Besides negative real numbers and complex numbers (which are ratios between vector, and not scalar, quantities), there are two specially problematic cases: 0 and 1. Zero cannot be the ratio between any two scalar quantities. The absence of a quantity is not itself a quantity. The real number *one*, if it existed, would exactly coincide with the relation of exact quantitative similarity. There are reasons for doubting that the latter is a real relation at all, as opposed to a merely logical or internal one. Hence, I would argue that it is best to think of 0 and 1 as useful fictions, rather than as constituents of the real world.

Thus, on this view, real numbers are relational tropes that are part of physical and other natural processes. As such, they can enter into causal interactions and explanations. The quantitative laws of nature, like the universal law of gravitation, connect the real-number tropes that measure mass and distance with those that measure the acceleration of motion. The axioms of real analysis are simply part of the theoretical superstructure of natural science.

3 Objection: The Absolute Generality of Mathematics

The sort of view that I've sketched briefly has, at least prima facie, some similarity to Mill's view of number. Consequently, it is vulnerable to Frege's well-known objection to that view, namely, that it fails to account for the absolute generality of mathematics. Let's keep our focus for the time being on the real numbers. Real numbers correspond, not only to ratios of physical quantities, but also to ratios of real numbers themselves. There may be number 3 tropes that represent one distance's being three times as great as another's, or one mass's being three times as great as another's, but there should also be number 3 tropes representing the number 3's being three times as great as the number 1. If real numbers are ratios, they must also be ratios among real numbers.

This was a decisive objection to Mill, since Mill tried to limit the ontological role of numbers to characterizing the physical world. I am imposing no such limitation, and neither did Bigelow, despite his misleading subtitle, "A Physicalist's Philosophy of Mathematics". Bigelow and I readily grant that real numbers can be used to measure the ratio of real numbers, as well as other mathematical objects. The problem we have to solve is this: when we think about ratios between real numbers, we seem to be talking about relations between universals and not tropes, and I have proposed identifying real numbers with relational tropes. In other words, it is not enough to explain what a number-3 trope is. We want to know what is *the* number 3, the unique real number 3.

Typically trope theorists have identified universals with classes of exactly similar tropes. Doing so in this context, however, leads to the threat of a vicious cycle. Suppose we identify the universal number 3 with the class of all number-3 tropes. Each of the number-3 tropes is borne by an ordered pair of quantities, in the first place, by ordered pairs of physical quantities. However, there must be a number-3 trope that is borne by the ordered pair $\langle 3, 1 \rangle$, where the members of this ordered pair are the *universals* 3 and 1. Thus, the class that is the universal 3 must include a member that is borne, in part, by the universal 3 itself. This seems unacceptably circular and ill-founded.

The best solution would seem to borrow from recent work in time and modality: specifically, David Lewis's counterpart theory. We should postulate that each number-3 trope is a counterpart of each of the others. The counterpart relation is simply the relation of exact quantitative similarity. We can then take assertions about universal numbers in one of two ways:

1. As ambiguously referring to any one of the counterparts.
2. As referring to a "dynamic" trans-type entity with counterparts as "stages".

If we take the first route, we can use a version of supervaluational semantics. A permissible interpretation of real analysis would consist of any complete and non-duplicative collection of real-number tropes.

If we take the second route, which I prefer, we should think of the universal number 3 as "growing" as we move up the hierarchy of types. Numerical universals are a kind of trans-type continuant, analogous to the trans-temporal, diachronic continuants of everyday life: persons and other organisms, for example.

Initially, the universal consists only of tropes borne by pairs of physical quantities. However, once universal real numbers are so constituted, they can themselves belong to pairs that bear further real-number tropes. These "new" real-number tropes are counterparts of the original ones, and so represent something like a new "stage" in the life-history of the universal numbers. The process continues up the type hierarchy. Whether it continues into the transfinite is an interesting question, one that I won't attempt to answer here.

4 Natural Numbers and Sets

As I mentioned earlier, we could think of arithmetic as simply a sub-theory of real analysis. However, there is a plausible, alternative strategy. Bigelow suggests that natural numbers are relations. In particular, the natural number 2 is that relation that hold between every pair of distinct entities. The number 2 is the universal irreflexive binary relation. Similarly, the number 3 is that relation that holds among any three distinct entities. This could obviously be generated to any cardinal number, including the transfinite ones.

Since I'm recommending that we approach things in terms of tropes instead of primitive universals, on my account for every two distinct entities, there is a trope of the number 2 kind that connects those two entities. Similarly for every triad, every quadrad, and so on. Here again, we face the problem of self-application. Among the things that can be numbered are the numbers themselves. Unlike Frege or Russell, I am not claiming that the number 2 is the class of all dyads or pair sets. Rather, I am claiming that the number 2 is a trans-type continuant, made up of dyadic tropes.

What about sets? The simplest hypothesis is to assert that all sets are cardinal-number tropes, and all cardinal-number tropes are sets. For example, for any two distinct entities a and b, there is a number 2 trope connecting them. We might as well identify this number 2 trope with the pair set $\{a, b\}$. The universal natural number 2 is a continuant made up of such pair sets. Alternatively, talk of the universal number 2 could be understood as referring indifferently to any of these pair sets.

This leads to a somewhat non-standard set theory, since I have to deny the real existence of the empty set, and of unit sets in general. I find this an advantage, rather than a disadvantage, since a *robust sense of reality*, of the kind recommend by Russell, should lead us to be skeptical of such dubious entities. We can treat the empty set and unit sets as useful fictions, while taking a fully realist attitude toward all other sets.

5 Is Mathematics Necessary or A Priori?

If it is a necessary truth that there exist at least two distinct entities that are not numbers, then I could use Plato's argument in the *Parmenides* to show that all the cardinal and ordinal numbers exist in every possible world. If a and b exist and are distinct, then there exists a number 2 trope. Since neither a nor b are numbers, there are then three distinct entities, and so there must be a number 3 trope. Each successive natural number is distinct from a and b and from all of its predecessors, and so the series of natural numbers is endless. The transfinite ordinals are constructible in a similar fashion. Hence, arithmetic and set theory consist entirely of necessary truths.

Real and complex analysis, in contrast, are only contingently true, since they depend on the existence of a sufficiently rich collection of physical quantities. These theories are, however, necessarily consistent.

On the question of a-priority, some of our mathematical knowledge may be innate, hard-wired into our neurology. Even if that's so, there is still a causal story to be told of how the physical instantiation of these numerical facts shaped the evolutionary development of our mathematical capacities.